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A Variation of Constants Formula and Bellman–Gronwall–Reid Inequalities

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1. There exist various generalizations of Bellman–Gronwall–Reid inequalities. We wish to prove, in this paper, a sufficiently general result in this direction which includes and unifies several recent works [1–4].

In Section 2, we develop a nonlinear variation of constants formula (2.5) for the scalar differential equation (2.1) under rather mild conditions. This is in itself interesting and may be used as a tool in applications. In Section 3, we give our main result on functional inequality and deduce some recent generalizations from our theorems.

2. In this section we wish to develop a variation of constants formula for the scalar differential equation

$$u' = \lambda(t)\, g(u) + R(t, u), \qquad u(t_0) = v_0 \geqslant 0, \tag{2.1}$$

where $\lambda \in C[I, R]$, $g \in C[R^+, R^+]$, $g(0) = 0$, $g(u) > 0$ for $u > 0$, $R \in C[I \times R^+, R]$ and $I = [t_0, T_0]$. It is well known that a solution $v(t)$ of

$$v' = \lambda(t)\, g(v), \qquad v(t_0) = v_0 \geqslant 0 \tag{2.2}$$

may be expressed in the form

$$G(v(t)) = \int_{t_0}^{t} \lambda(s)\, ds + G(v_0), \qquad t \in I_0, \tag{2.3}$$

where

$$G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}, \qquad u_0, u > 0$$

and

$$I_0 = \left[t \in I : G(v_0) + \int_{t_0}^{t} \lambda(s)\, ds \in \mathrm{dom}(G^{-1})\right].$$

Let us now apply the variation of constants method. For this we must be able to determine v_0 as a function t such that $v(t)$ is a solution of (2.1). To this end, we find

$$G_u(v(t))\, v'(t) = \lambda(t) + G_u(v_0)\, v_0',$$

which, in view of the definition of G and (2.1) reduces to

$$v_0' = \frac{g(v_0)}{g(v(t))} R(t, v(t)).$$

Using now (2.3) to eliminate $v(t)$, we obtain the differential equation

$$v_0' = w(t, v_0), \qquad v_0(t_0) = v_0\,, \tag{2.4}$$

where

$$w(t, v_0) = \frac{g(v_0)\, R\left(t, G^{-1}\left(G(v_0) + \int_{t_0}^{t} \lambda(s)\, ds\right)\right)}{g\left[G^{-1}\left(G(v_0) + \int_{t_0}^{t} \lambda(s)\, ds\right)\right]}.$$

Let $v_0(t)$ be a solution of (2.4) existing on I_0. Then (2.3) gives the integral equation satisfied by a solution $v(t)$ of (2.1) in the form, for $t \in I_0$,

$$\begin{aligned} v(t) = G^{-1}\Bigg[\int_{t_0}^{t} \lambda(s)\, ds \\ + G\left(v_0 + \int_{t_0}^{t} \frac{g\left\{G^{-1}\left(G(v(s)) - \int_{t_0}^{s} \lambda(\xi)\, d\xi\right)\right\} R(s, v(s))}{g(v(s))}\right)\Bigg]. \end{aligned} \tag{2.5}$$

We have proved the following

THEOREM 2.1. *Let λ, g, R and I be as given above. Then a solution $v(t)$ of* (2.1) *can be exhibited in the form* (2.5) *on I_0, and it can be obtained by the method of variation of constants.*

We note, in passing, that the linear case is covered by (2.5).

COROLLARY 2.1. *If, in addition to the assumptions of Theorem* 2.1, *we assume that $\lambda \geqslant 0$, $R(t, u) = \sigma(t) \geqslant 0$ and g is nondecreasing in u, then the following bound is true for $v(t)$:*

$$v(t) \leqslant G^{-1}\left[\int_{t_0}^{t} \lambda(s)\, ds + G\left(v_0 + \int_{t_0}^{t} \sigma(s)\, ds\right)\right], \qquad t \in I_0\,.$$

3. Here we shall consider the functional inequality, for $t \in I$,

$$f(x(t)) \leqslant a(t) + b(t) + h\left[c(t) + \int_{t_0}^{t} k(t, s)\, w(s, x(s))\, ds\right], \tag{3.1}$$

under a variety of conditions on h.

THEOREM 3.1. *Let x, a, b, $c \in C[I, R^+]$, f, $h \in C[R^+, R^+]$, f be strictly increasing, g be nondecreasing, $k \in C[I \times I, R^+]$, $w \in C[I + R^+, R^+]$ and $w(t, u)$ be nondecreasing in u for each t. Define*

$$A(t) = \max_{t_0 \leqslant s \leqslant t} a(s), \qquad B(t) = \max_{t_0 \leqslant s \leqslant t} b(s), \qquad C(t) = \max_{t_0 \leqslant s \leqslant t} c(s)$$

and

$$K(t, s) = \max_{s \leqslant \sigma \leqslant t} k(\sigma, s).$$

Then

(i) $$x(t) \leqslant f^{-1}[a(t) + b(t)\, h\{r_1(t, t_0, C(t))\}], \qquad t \in I_{10}, \tag{3.2}$$

where $r_1(T, t_0, r_{10})$ is the maximal solution of

$$r_1' = K(T, t)\, w[t, f^{-1}(a(t) + b(t)\, h(r_1))], \qquad r_1(t_0) = r_{10},$$

existing on $I_1 \subset I$.

(ii) *If, in addition, $h_u(u)$ exists, continuous and nondecreasing in u, then*

$$x(t) \leqslant f^{-1}[r_2(t, t_0, A(t) + B(t)\, h(C(t)))], \qquad t \in I_{20}, \tag{3.4}$$

where $r_2(T, t_0, r_{20})$ is the maximal solution of

$$r_2' = B(T)\, h_u\left[h^{-1}\left(\frac{r_2 - A(T)}{B(T)}\right)\right] K(T, t)\, w(t, f^{-1}(r_2)), \qquad r_2(t_0) = r_{20}, \tag{3.5}$$

existing on $I_1 \subset I$.

(iii) *If, $h^{-1}(u)$ in convex, submultiplicative and $\alpha, \beta > 0$, continuous on I such that $\alpha(t) + \beta(t) = 1$, then the following two types of estimates are valid:*

(iiia) $$x(t) \leqslant F^{-1}[r_3(t, t_0, C(t))], \qquad t \in I_{30}, \tag{3.6a}$$

where $r_3(T, t_0, r_{30})$ is the maximal solution of

$$r_3' = K(T, t)\, w[t, F^{-1}(m(t) + n(t)\, r_3)], \qquad r_3(t_0) = r_{30}, \tag{3.7a}$$

existing on $I_0 \subset I$. Here

$$F = h^{-1} \cdot f, \qquad m(t) = \alpha(t)\, h^{-1}[a(t)\, \alpha(t)^{-1}]$$

and

$$n(t) = \beta(t)\, h^{-1}[b(t)\, \beta(t)^{-1}];$$

(iiib) $$x(t) \leqslant F^{-1}[r_4(t, t_0, M(t) + N(t)\, C(t))], \qquad t \in I_{40}, \tag{3.6b}$$

where $r_4(T, t_0, r_{40})$ *is the maximal solution of*

$$r_4' = N(T)\, K(T, t)\, w(t, F^{-1}(r_4)), \qquad r_4(t_0) = r_{40}, \tag{3.7b}$$

existing on $I_0 \subset I$. *Here* F *is as in* (iiia),

$$M(t) = \max_{t_0 \leqslant s \leqslant t} m(s) \qquad \textit{and} \qquad N(t) = \max_{t_0 \leqslant s \leqslant t} n(s).$$

In each of the foregoing cases, I_{i0} $i = 1, 2, 3, 4$, is the appropriate interval contained in I subject to the domains of the inverse functions involved.

Proof. (i) We observe that for $t_0 \leqslant t \leqslant T \leqslant T_0$, we have from (3.1),

$$f(x(t)) \leqslant a(t) + b(t)\, h[v(t, T)], \tag{3.8}$$

where

$$v = v(t, T) = C(T) + \int_{t_0}^{t} K(T, t)\, w(s, x(s))\, ds. \tag{3.9}$$

Consequently, it follows that

$$v' \leqslant K(T, t)\, w[t, f^{-1}\{a(t) + b(t)\, h(v)\}].$$

By Theorem 1.4.1 in [4], we readily get

$$v(T, T) \leqslant r_1(T, t_0, v(t_0, T)), \qquad T \geqslant t_0,$$

where $r_1(T, t_0, r_{10})$ is the maximal solution of (3.3) with $r_{10} = v(t_0, T)$. By (3.8) and (3.9), there results

$$f(x(T)) \leqslant a(T) + b(T)\, h[r_1(T, t_0, C(T))], \qquad T \geqslant t_0, \tag{3.10}$$

which implies the stated estimate (3.2).

(ii) In this case, we may write (3.1), for $t_0 \leqslant t \leqslant T \leqslant T_0$, as

$$f(x(t)) \leqslant A(T) + B(T)\, h(v), \tag{3.11}$$

where $v = v(t, T)$ is the same function defined in (i). Setting

$$z = z(t, T) = A(T) + B(T)\, h(v),$$

it is easy to get

$$z' \leqslant B(T)\, h_u \left[h^{-1}\left(\frac{z - A(T)}{B(T)}\right)\right] K(T, t)\, w(t, f^{-1}(z)). \tag{3.12}$$

Again Theorem 1.4.1 in [4], yields

$$z(T, T) \leqslant r_2(T, t_0, z(t_0, T)), \qquad T \geqslant t_0, \tag{3.13}$$

where $r_2(T, t_0, r_{20})$ is the maximal solution of (3.5) with

$$r_{20} = A(T) + B(t)\, h(C(T)).$$

By (3.11) and (3.13), the bound (3.4) results because of the definition of z, arguing as before.

(iii) The inequality (3.1) can be written as

$$\begin{aligned} h^{-1} \cdot f(x(t)) &\leqslant \alpha(t)\, h^{-1}(a(t)\, \alpha(t)^{-1}) \\ &\quad + \beta(t)\, h^{-1}(b(t)\, \beta(t)^{-1}) \left[c(t) + \int_{t_0}^{t} k(t, s)\, w(s, x(s))\, ds\right]; \end{aligned}$$

using the convexity and submultiplicity of h^{-1}. This reduces, because of the definition of f, m and n, to

$$F(x(t)) \leqslant m(t) + n(t) \left[c(t) + \int_{t_0}^{t} k(t, s)\, w(s, x(s))\, ds\right]. \tag{3.14}$$

If we treat (3.14) as a special case of (i) with $h(u) = u$, $F = f$, $a = m$ and $b = n$, we arrive at (3.6a) and (3.7a) from (3.2) and (3.3) under appropriate substitutions. If, on the other hand, we treat (3.14) as a particular case of (ii), we get (3.6b) and (3.7b) from (3.4) and (3.5), respectively.

The proof of the theorem is complete.

Employing our theorems, we shall show that the recent results in [1–4] can be derived as special cases. First of all to get the result of Gollwitzer [1], take $f(u) = u$, $c(t) = 0$, $k(t, s) = 1$, $w(t, u) = \lambda(t)\, g(u)$ and $h = g^{-1}$. Then it follows by Theorem 3.1(iiia) that

$$x(t) \leqslant g^{-1}[m(t) + n(t)\, r_3(t, t_0, 0)], \qquad t \in I,$$

where

$$r_3(t, t_0, 0) = \int_{t_0}^{t} \exp\left[\int_{s}^{t} \lambda(\xi)\, n(\xi)\, d\xi\right] \lambda(s)\, m(s)\, ds.$$

Also, by Theorem 3.1(iiib), one could get another estimate

$$x(t) \leqslant g^{-1}\left[M(t) \exp\left(\int_{t_0}^{t} N(t)\, \lambda(s)\, ds\right)\right], \qquad t \in I.$$

The work of Butler and Rogers [2] follows from ours on noting that $h(u) = u$, $c(t) = 0$ and $w(t, u) = g(u)$. We then get by Theorem 3.1(ii), the relation

$$x(t) \leqslant f^{-1}[r_2(t, t_0, A(t))],$$

where

$$r_2(T, t_0, A(T)) = \Omega^{-1}\left(\Omega(A(T)) + \int_{t_0}^{t} B(T)\, K(T, s)\, ds\right),$$

Ω being given by

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{g \cdot f^{-1}(s)}, \qquad u, u_0 > 0.$$

Next we derive the result of Deo and Murdeshwar [3]. Choose $f(u) = u$, $b(t) = 1$, $k(t, s) = 1$ and $w(t, u) = \lambda(t)\, g(u)$, $g(u)$ being subadditive. Then Theorem 3.1(i) yields

$$x(t) \leqslant a(t) + h(r_1(t, t_0, 0)),$$

where $r_1(t, t_0, 0)$ is the maximal solution of

$$r_1' = \lambda(t)\, H(r_1) + \sigma(t)$$

with $H = g \cdot h$ and $\sigma(t) = \lambda(t)\, g(a(t))$. Thus, by Corollary 2.1, noting that H is nondecreasing, we arrive at

$$r_1(t, t_0, 0) \leqslant G^{-1}\left[\int_{t_0}^{t} \lambda(s)\, ds + G\left(\int_{t_0}^{t} \sigma(s)\, ds\right)\right] \equiv r(t, t_0, 0),$$

where

$$G(u) = \int_{u_0}^{u} \frac{ds}{H(s)}, \qquad u, u_0 > 0.$$

We therefore obtain

$$x(t) \leqslant a(t) + h(r(t, t_0, 0)).$$

Finally, suppose we let $f(u) = u$, $b(t) = 1$, $h(u) = u$, $c(t) = 0$ and $k(t, s) = 1$, then we deduce Corollary 1.9.4 in [4] from (3.2) which is a generalization of Bellman–Gronwall–Reid inequality in a rather general form. Clearly, for various choices of the functions involved in (3.1), one can derive from our results respective explicit bounds. We do not go into further details.

References

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